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A NOTE ON THE TRANSMISSION OF OBLIQUE WAVES THROUGH SMALL APERTURES

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Abstract—The transmission of water waves obliquely incident on a thin vertical wall with a submerged narrow horizontal gap in deep water is considered by employing Havelock's expansion of the water wave potential. The approximate expression for the transmission coefficient is obtained.

1. INTRODUCTION

This note is concerned with the problem of oblique wave incidence on a vertical wall with a submerged narrow horizontal gap. This problem has been considered in some detail by Liu and Wu [5,6], who obtained the solutions using the method of matched asymptotic expansion similar to Tuck [9] for the case of normal incidence, and results were presented for both finite and infinite depth of water and for thin as well as thick walls. In [5,6], while the far field expansion of the wave potential is correct, the near field expansion seems to be valid only for small wave numbers since the governing equation (Helmholtz's equation) is approximated by Laplace's equation. Mandal [7] recently used the method of Havelock's [3] expansion of water wave potential to attack the normally incident wave problem of Tuck [9] resulting in an integral equation which was solved by using a technique given by Packham and Williams [8] that is valid when the gap is narrow, and the approximate expression of the transmission coefficient was then obtained.

In the present note, the problem of oblique wave incidence on a thin wall with a narrow gap is reinvestigated by using the modified Havelock's expansion theorem (cf. Green [2]) for the water wave potential satisfying the Helmholtz's equation. This results in an integral equation which is solved approximately employing the method described in [8] and approximate expression for the transmission coefficient is then calculated. For normal incidence of the wave train, this expression reduces to the result derived by Tuck [9].

The problem considered here seems to have applications in mathematical modelling associated with the construction of structures known as breakwaters to protect a port from the impact of the rough sea during a storm. In fact, at the Suma Port and Tarumi Fishing Port in Japan, perforated breakwaters have been constructed for the protection of the sheltered area of the ports by reflecting incident waves from the sea (cf. Koh [4]). The simplest type of perforated breakwater is made of concrete caissons with horizontal rectangular slots [6]. A thin vertical wall with a gap is perhaps the simplest model of such a breakwater.

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2. FORMULATION OF THE PROBLEM

We consider the transmission of a train of surface water waves obliquely incident on a fixed thin-plane vertical wall extending from above the free surface to infinitely downwards but having a horizontal gap of length $2a$, which is small compared to depth h of its center below the mean-free surface. The situation under consideration is sketched in Figure 1. We use a coordinate system in which the y-axis is taken vertically downwards, the mean-free surface is the xz-plane, the wall is the yz-plane, and the gap occupies the position $x = 0$, $h-a < y < h+a$, $-\infty < z < \infty$. Assuming water to be homogeneous, inviscid, incompressible and the motion in the water to be irrotational and due to the obliquely incident train of time harmonic surface waves of angular frequency σ, and of small amplitude, the fluid motion can be described by a velocity potential which is the real part of $\chi(x,y,z)\exp(-i\sigma t)$. $\chi^{\text{inc}} = \exp(-Ky + i\mu x + i\nu z)$, $\mu = K\cos\alpha$, $\nu = K\sin\alpha$ is assumed to be incident at an angle α to the normal of the wall from negative infinity. Such a wave train will be partially reflected by the wall and partially transmitted through the gap, and in view of the geometry of the wall we can assume $\chi(x,y,z) = \varphi(x,y)\exp(i\nu z)$. Then $\varphi(x,y)$ must satisfy

$$\varphi_{xx} + \varphi_{yy} - \nu^2\varphi = 0 \tag{2.1}$$

in the fluid region,

$$K\varphi + \varphi_y = 0 \text{ on } y = 0 \tag{2.2}$$

with $K = \sigma^2/g$, g being the gravity.

$$\varphi_x = 0 \text{ on } x = \pm, \quad 0 < y < h-a, \quad h+a < y < \infty, \tag{2.3}$$

$$\nabla\varphi \to 0 \text{ as } y \to \infty, \tag{2.4}$$

φ and its derivatives are continuous everywhere except possibly across $x = 0$, $0 < y < h-a$, $h+a < y < \infty$. We also require that φ and its derivatives are bounded everywhere away from the two sharp edges of the wall at the gap and near these edges $r^{1/2}\nabla\varphi$ is bounded, where r is the distance from either of the edges. Finally, we assume that $\varphi(x,y)$ has the asymptotic forms

$$\left.\begin{aligned} \varphi(x,y) &\sim T\exp(-Ky + i\mu x) && \text{as } x \to \infty, \\ \varphi(x,y) &\sim \exp(-Ky + i\mu x) + R\exp(-Ky - i\mu x) && \text{as } x \to -\infty \end{aligned}\right\} \tag{2.5}$$

where T and R are the complex transmission and reflection coefficients. Our problem is to determine T and R.

Using the modified Havelock's expansion (cf. [2]) for the water wave potential satisfying the Helmholtz's equation and noting (2.5) we can write

$$\begin{aligned} \varphi(x,y) &= T\exp(-Ky + i\mu x) \\ &\quad + \int_0^\infty A(k)(k\cos ky - K\sin ky)\exp[-(k^2+\nu^2)^{1/2}x]dk, \quad x > 0. \\ &= \exp(-Ky + i\mu x) + R\exp(-Ky - i\mu x) \\ &\quad + \int_0^\infty B(k)(k\cos ky - K\sin ky)\exp[(k^2+\nu^2)^{1/2}x]dk, \quad x < 0. \end{aligned} \tag{2.6}$$

If we assume

$$\varphi_x(\pm 0, y) = f(y), 0 < y < \infty, \tag{2.7}$$

then in view of (2.3),

$$f(y) = 0 \text{ for } 0 < y < h-a, h+a < y < \infty \tag{2.8}$$

and also

$$f(y) = \begin{cases} 0((y-h+a)^{-1/2}) & \text{as } y \to h-a, \\ 0((h+a-y)^{-1/2}) & \text{as } y \to h+a. \end{cases} \tag{2.9}$$

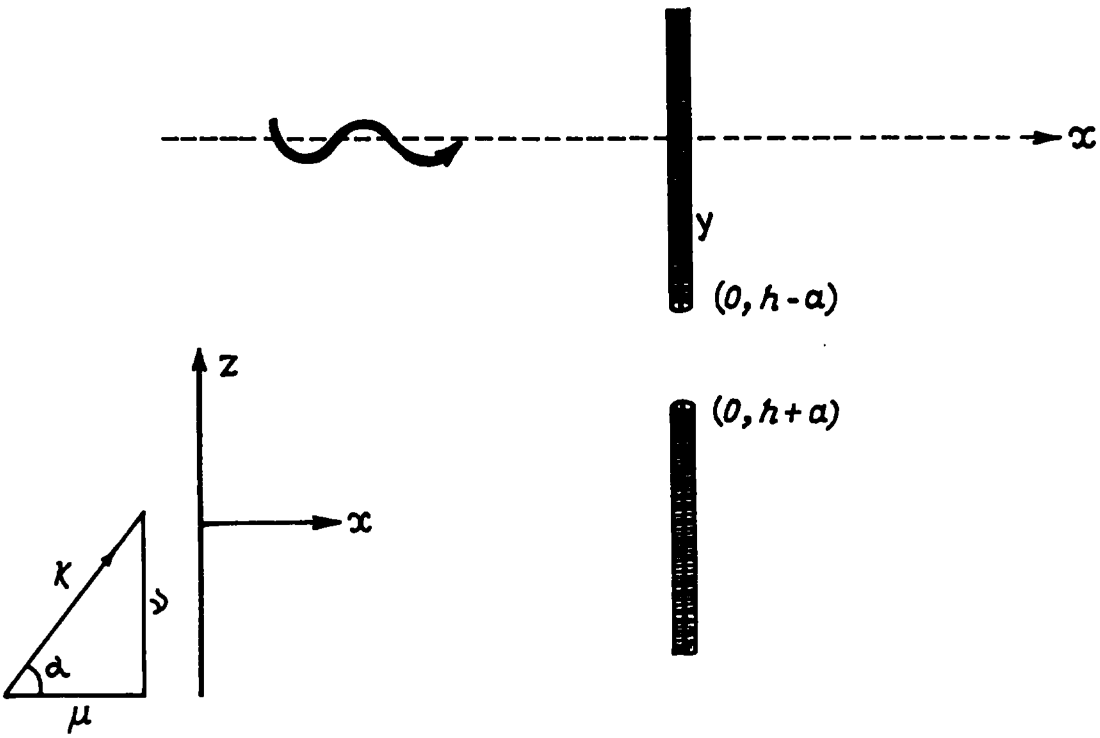

Figure 1.

Using (2.7) in (2.6), we obtain for all y

$$\left.\begin{aligned} f(y) &= i\mu T\exp(-Ky) - \int_o^\infty A(k)(k^2+\nu^2)^{1/2}(k\cos ky - K\sin ky)dk \\ &= i\mu(1-R)\exp(-Ky) + \int_o^\infty B(k)(k^2+\nu^2)^{1/2}(k\cos ky - K\sin ky)dk. \end{aligned}\right\} \tag{2.10}$$

Hence by Havelock's expansion theorem (cf. [2])

$$\left.\begin{aligned} i\mu T = i\mu(1-R) &= 2K\int_o^\infty f(y)\exp(-Ky)dy \\ \text{and} \qquad -(k^2+\nu^2)^{1/2}A(k) &= (k^2+\nu^2)^{1/2}B(k) \\ &= \frac{2}{\pi(k^2+K^2)}\int_o^\infty f(y)(k\cos ky - K\sin ky)dy. \end{aligned}\right\} \tag{2.11}$$

Utilizing the condition that $\varphi(x,y)$ is continuous across the gap $x=0$, $h-a<y<h+a$ we obtain

$$T\exp(-Ky) + \int_o^\infty A(k)(k\cos ky - K\sin ky)dk =$$
$$(1+R)\exp(-Ky) + \int_o^\infty B(k)(k\cos ky - K\sin ky)dk, |y-h|<a.$$

Using (2.11) and (2.8), this gives the integral equation

$$\frac{1}{\pi}\int_{h-a}^{h+a} f(u)\left[\int_o^\infty \frac{2(k\cos ky - K\sin ky)(k\cos ku - K\sin ku)}{(k^2+\nu^2)^{1/2}(k^2+K^2)}dk\right]du$$
$$= (T-1)\exp(-Ky), h-a<y<h+a. \tag{2.12}$$

This integral equation is solved approximately by exploiting the concept of narrowness of the gap.

The inner integral of (2.12) can be simplified and is equivalent to

$$K_o(\nu|y-u|) - K_o(\nu|y+u|) + 2\int_\nu^\infty \frac{k\exp[-k(y+u)]}{(k^2-\nu^2)^{1/2}(k-K)}dk \tag{2.13}$$

where the integral is in the sense of Cauchy principal value. Now,

$$K_o(\nu|y-u|) \sim \ell n\left(\frac{2}{\nu|y-u|}\right) \text{ as } |y-u| \to 0. \tag{2.14}$$

Thus in view of (2.13) and (2.14), the integral equation (2.12) can be written as

$$\int_{h-a}^{h+a} f(u)\ell n|y-u|du = \pi(1-T)\exp(-Ky) + \int_{h-a}^{h+a} f(u)g(y,u)du \tag{2.15}$$

where $g(y,u)$ is a regular function. Since for a narrow gap $a \ll h$, y and u in $\exp(-Ky)$ and $g(y,u)$ in the right-hand side of (2.15) is replaced by h. Thus the integral equation (2.15) is approximated as

$$\int_{h-a}^{h+a} f(u)\ell n|y-u|du = c. \tag{2.16}$$

where

$$\left.\begin{array}{rcl} c &=& \pi(1-T)\exp(-Kh) + g(h,h)\lambda, \\ g(h,h) &=& \ell n2/\nu - K_o(2\nu h) + 2\int_\nu^\infty \frac{k\exp(-2kh)}{(k^2-\nu^2)^{1/2}}\frac{dk}{k-K}, \\ \lambda &=& \int_{h-a}^{h+a} f(u)du. \end{array}\right\} \tag{2.17}$$

Following Cooke [1] the solution of (2.16) satisfying (2.9) is

$$f(y) = \frac{1}{[a^2-(y-h)^2]^{1/2}}\frac{c}{\pi\ell n\frac{a}{2}} \tag{2.18}$$

provided $\frac{a}{2} \neq 1$. The unknown constant λ satisfies

$$\lambda = \frac{\pi(1-T)\exp(-Kh) + g(h,h)\lambda}{\ell n\frac{a}{2}}$$

so that it is given by

$$\lambda = \pi(1-T)\exp(-Kh)/\left[\ell n\left(\frac{a}{2}\right) - g(h,h)\right]. \tag{2.19}$$

3. TRANSMISSION COEFFICIENT

Utilizing (2.11), we find that

$$\begin{aligned} T &= -2i\sec\alpha\int_{h-a}^{h+a} f(y)\exp(-Ky)dy \\ &\simeq -2\pi i(1-T)\sec\alpha\exp(-Kh)/\left[\ell n\frac{a}{2} - g(h,h)\right] \end{aligned}$$

so that we finally obtain

$$\begin{aligned} T = -\ i\sec\alpha/&\left[\frac{\exp(2Kh)}{2\pi}\left\{\ell n\left(\frac{Ka\sin\alpha}{4}\right) + K_o(2Kh\sin\alpha)\right\}\right. \\ &\left. + \frac{1}{\pi}F(Kh;\alpha) - i\sec\alpha\right] \end{aligned} \tag{3.1}$$

where

$$F(Kh;\alpha) = -\exp(2Kh) f_{Kh\sin\alpha}^{\infty} \frac{k\exp(-2kh)}{(k^2 - K^2\sin^2\alpha)^{1/2}} \frac{dk}{k-K}. \tag{3.2}$$

The reflection coefficient is obtained immediately by using (2.11) as $R = 1 - T$.

As a check to the result (3.2), we now make $\alpha \to 0$ (the case of normal incidence). Since

$$\lim_{a\to 0}\left[\ell n\left(\frac{Ka\sin\alpha}{4}\right) + K_o(2Kh\sin\alpha)\right] = \ell n\frac{a}{4h}$$

and

$$\lim_{a\to 0} F(Kh;\alpha) = \bar{E}_i(2Kh)$$

we obtain

$$T_o = \lim_{\alpha\to 0} T = -i/\left[\frac{\exp(2Kh)}{2\pi}\ell n\frac{a}{4h} + \frac{1}{\pi}\bar{E}_i(2Kh) - i\right]$$

and this coincides with Tuck's result (the sign of i is different here as we have taken the time dependence to be $\exp(-i\sigma t)$).

4. DISCUSSION

An approximate expression for the transmission coefficient for an obliquely incident wave train through a narrow gap in a thin vertical wall is obtained here by using the technique of Havelock's expansion of water wave potential leading to the solution of an integral equation. For $\alpha = 0^o$, the expression (3.1) reduces to the result already obtained by Tuck [9] for the normal incidence case by using the method of matched asymptotic expansion, while for $\alpha = 90^o$, T becomes 1 which is expected since in this case the wave train is fully transmitted as the incident waves propagate parallel to the wall.

The transmission coefficient $|T|$ is evaluated numerically for various values of the incident angle α, wave number K and the ratio $2a/h$ of the thickness of the gap with its depth. Some typical calculations for $|T|$ are given in Table 1 for $\alpha = 45^o$ and $\frac{2a}{h} = 0.05$. From Table 1, it is observed that $|T|$ first increases and then decreases asymptotically to zero with the increase of Kh from 0.1. For other fixed values of α and $2a/h$ similar qualitative behavior is observed. This is expected since when Kh becomes large the wave train is confined near the free surface and is almost reflected totally by the part of the wall above the gap.

Again, for fixed Kh and $2a/h$, it is observed that $|T|$ first decreases as α increases from 0^o and then increases for further increase of α. A typical numerical illustration is given in Table 2. This implies that up to a certain value of the angle of incidence (in our calculation it is nearly 5^o), the energy transmission decreases and beyond this angle it increases monotonically with α up to 90^o and takes the value 1 for $\alpha = 90^o$.

Table 1. $\alpha = 45^o$, $2a/h = 0.05$

Kh	0.1	0.2	0.3	0.5	1.0	1.5	2.0
$\lvert T\rvert$	0.7294	0.7413	0.7169	0.6088	0.2707	0.0995	0.0366

Table 2. $Kh = 0.5$, $2a/h = 0.05$

α	0^o	5^o	10^o	15^o	30^o	60^o	85^o
$\lvert T\rvert$	0.5318	0.4763	0.4803	0.4877	0.5302	0.7345	0.9830

As noted in the introduction, in the technique of matched asymptotic expansion used in [5,6], while the far-field solution is correct, the near-field solution is valid only for small wave numbers or for small angle of incidence as the governing equation (2.1) which is a two-dimensional modified Helmholtz's equation, is approximated by the two-dimensional Laplace's equation. Thus the approximate expression for the transmission coefficient (for the thin barrier problem) given in [5, eq. 38], as well as the other results obtained in [5,6] for wide barrier, are not valid for all angle of incidence of the wave train and all wave numbers.

To obtain a near-field solution which is valid for all angles of incidence and wave numbers, it is necessary to solve the two-dimensional modified Helmholtz's equation in an infinite medium in the presence of a barrier with a slit. This problem needs special attention for solution.

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